Int. J. Solids Structures Vol. 25, No. 8, pp. 863–877, 1989
Printed in Great Britain.
0020-7683/89 $3.00 + .00

A ONE-DIMENSIONAL STUDY OF THE LIMIT CASES OF THE ENDOCHRONIC THEORY

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(*Received* 16 *October* 1987; *in revised form* 6 *June* 1988)

Abstract—A simple isothermic one-dimensional endochronic model is constructed to analyse the stress-response using the various intrinsic time-scales proposed by Valanis. It is shown that the use of an intrinsic time-scale defined in terms of the plastic strain, which represents a limit case of the endochronic theory according to Valanis, leads to an elastic law and not to a conventional plasticity theory, as claimed in that reference. This apparent paradox will be revealed to hold for the most general form of a one-dimensional endochronic model. It will be demonstrated, within this context, that the evolution equations of the model degenerate into a constraint condition impeding all plastic deformation, when an intrinsic time-scale based on the plastic strain is used. Finally, it will be proved that no one-dimensional endochronic model can ever predict an elastic unloading following previous loading in which plastic flow has occurred, and thereby describe the material behaviour of metals as done by conventional plasticity theories.

1. INTRODUCTION

The description of rate-independent material-behaviour by means of a strain-path parametrization dates back to Il'yushin (1954). This concept was treated within the framework of the functional theory of simple materials with memory by Pipkin and Rivlin (1965). The representation form at which these authors arrive was shown by Owen and Williams (1968) to be equivalent to the definition of rate-independency put forward by Truesdell and Noll (1965), which the latter authors had used within the context of hypoelasticity.

It was not, however, until 1971 that an attempt was finally made to apply Il'yushin's concept to the description of rate-independent material behaviour, such as characterizing elastoplastic deformations. This was carried out by Valanis (1971a) starting from a visco-elastic theory with inner variables advanced by the same author in Valanis (1966), in which the time is formally replaced by a material-dependent time-transformation defined in terms of the total strain-path. This gave birth to the original version of the endochronic theory (OVET), which enjoys the following advantageous properties:

(a) The stress-response to a given strain-history is explicitly determined by a functional of the latter;

(b) The theory prescinds from a loading criterion, i.e. the aforementioned stress-response is always described by the same set of constitutive equations;

(c) No decomposition whatsoever of the strain measure is required for the theory.

Another essential, though not necessarily advantageous feature of the OVET consists in avoiding the use of any kind of yield criteria. Such a feature, while appealing from the point of view of the implementation of the theory, is not to be looked upon as advantageous inasmuch as the lack of a yield criterion renders impossible the description of any finite elastic range, as was pointed out by Rivlin (1981). In this connection, one may regard the OVET as a plasticity theory whose yield surface has been shrunk to a point (namely the origin of the stress-space).

The OVET proved to be capable of predicting various aspects of the mechanical behaviour of metals, such as the effect of cross-hardening (see for instance Valanis, 1971b, 1974; Valanis and Wu, 1975), but fails to correctly describe the stress-response of such materials to deformation processes involving reversal-points in the loading (as in the case of cyclic deformation-processes) (see Valanis, 1975; Rivlin, 1981). Such a shortcoming of the theory *vis-à-vis* the real mechanical behaviour of metals obviously entails a substantial restriction on its field of application. Aiming at the correction of this deficiency, a new version of the endochronic theory was developed in Valanis (1980) by generalizing the concept of strain-path used in the theory. In that reference, the strain-path is in such a way defined so as to encompass as limit cases the strain-path used in the OVET (i.e. the total strain-path), as well as that traditionally employed in conventional plasticity theories, namely the familiar plastic strain-path. The generalized strain-path is then used to formally replace the original strain-path entering the constitutive equations of the OVET. According to Valanis' ensuing analysis of the stress-response resulting from this substitution, the generalization of the strain-path leads to an improved version of the endochronic theory (IVET) in the sense that for the limit case, in which the generalized strain-path coincides with the plastic strain, the theory enjoys the following desirable properties not exhibited by the OVET:

(a) The IVET is capable of predicting a stress-response to deformation processes including reversal-points in loading in agreement with the experimentally observed mechanical behaviour of metals;

(b) The well-known von Mises yield-condition is derived from the constitutive equations of the IVET, thus enabling the description of a finite elastic range;

(c) A loading criterion ensues from the constitutive equations of the IVET.

Moreover, Valanis concludes from his analysis of the IVET that conventional plasticity theories are to be regarded as a special case of the former, corresponding to the limit case above.

The object of this paper is to discuss these claims *vis-à-vis* the behaviour of a one-dimensional specialization of the IVET.

2. THE ONE-DIMENSIONAL SPECIALIZATION OF THE ENDOCHRONIC THEORY

We will begin our discussion by constructing a one-dimensional model of the endochronic theory, which shall be general enough to encompass the OVET as well as the IVET as special cases. Our attention shall be restricted, as in the above references, to small deformations. Let σ be the stress, ε the strain, θ the absolute temperature, ρ the mass density, x_α a set of n inner variables. We assume the stress-response to be given by

$$\sigma = E_0\varepsilon + \rho \sum_{\alpha=1}^{n} B_\alpha x_\alpha, \quad E_0 > 0, \quad B_\alpha \geqslant 0 \tag{1}$$

where E_0 is the Young modulus and B_α are material parameters. We assume furthermore that the behaviour of x_α is described by the following set of first-order differential equations,

$$\frac{d\hat{x}_\alpha}{dz} = \Omega_\alpha(\hat{\varepsilon}, \hat{\theta}, \hat{x}_\beta), \quad \alpha, \beta = 1, 2, \ldots, n \tag{2}$$

which will be referred to as the "endochronic evolution equations" throughout this work. The hat therein is used in (2) and hereafter as a descriptive mark for the composition of any function of the time t with

$$t = g^i_{\varepsilon_i}(z) = \min\{t \mid g_{\varepsilon_i}(t) = z\} \tag{3}$$

where $z = g_{\varepsilon_i}(t)$ will be defined below and i denotes the generalized inverse function in the sense of Owen and Williams (1968). Functions denoted with a hat will be said to be endochronic parametrized in what follows.

Following Valanis (1980), we define a generalized inelastic strain ε_i by

$$\mathrm{d}\varepsilon_i = \mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma = (1-\kappa)\,\mathrm{d}\varepsilon - \kappa\,\mathrm{d}\varepsilon_p, \quad 0 \leqslant \kappa \leqslant 1 \tag{4}$$

where κ is a dimensionless parameter and ε_p is the plastic strain defined by

$$\mathrm{d}\varepsilon_p = \mathrm{d}\varepsilon - \frac{1}{E_0}\mathrm{d}\sigma. \tag{5}$$

We now define the generalized strain path ζ by

$$\zeta = l_{\varepsilon_i}(t) = \int_0^t P(\varepsilon_i)|\dot{\varepsilon}_i|\,\mathrm{d}t \Rightarrow \mathrm{d}\zeta = P(\varepsilon_i)|\mathrm{d}\varepsilon_i| \tag{6}$$

where $P(\varepsilon_i)$ is a non-negative dimensionless material function which plays the role of a metric.

Following Valanis (1980), we introduce the intrinsic time z defined by

$$z = \omega(\zeta) = \int_0^\zeta \frac{1}{f(\zeta')}\mathrm{d}\zeta' \Rightarrow \mathrm{d}z = \frac{\mathrm{d}\zeta}{f(\zeta)} \tag{7}$$

where $f(\zeta)$ is a dimensionless continuous positive monotone non-decreasing material function.

The composition of ω and l_{ε_i} is then called the intrinsic time scale, i.e.

$$z = g_{\varepsilon_i}(t) = \omega\, 0\, l_{\varepsilon_i}(t) = \omega(l_{\varepsilon_i}(t)). \tag{8}$$

We remark that unless ε_i remains constant for some period of time, $t = g_{\varepsilon_i}^{-1}(z)$ exists, which is then identical with $t = g^{i}_{\varepsilon_i}(z)$ (the latter being in either case defined) since

$$|\dot{\varepsilon}_i| > 0 \Leftrightarrow \dot{z} > 0. \tag{9}$$

We now restrict our attention to isothermic deformations. In particular, we assume the endochronic evolution equations to be uncoupled and linear with constant coefficients

$$\eta_\alpha \hat{\varepsilon} + \lambda_\alpha \hat{x}_\alpha + \frac{\mathrm{d}\hat{x}_\alpha}{\mathrm{d}z} = 0, \quad \eta_\alpha > 0, \quad \lambda_\alpha > 0 \tag{10}$$

where η_α, λ_α are positive material parameters.

Under the initial condition $x_\alpha(0) = 0$ the solution of (10) may be obtained and then replaced in (1), thus yielding the following explicit stress-response,

$$\sigma = E_0\varepsilon - \int_0^z \sum_{\alpha=1}^{n} \lambda_\alpha E_\alpha\, \mathrm{e}^{-\lambda_\alpha(z-z')}\hat{\varepsilon}(z')\,\mathrm{d}z' \tag{11}$$

where E_α stands for a material parameter having the dimension of a stress which is defined by

$$E_\alpha = \rho\frac{\eta_\alpha}{\lambda_\alpha}B_\alpha > 0. \tag{12}$$

Taking the initial condition $\hat{\varepsilon}(0) = 0$ into account, we may integrate (11) by parts and subsequently draw the free terms under the integral. It should be borne in mind, that this last step entails the endochron parametrization of the free terms, which in turn presupposes that $z > 0$. We arrive thus at

$$\sigma = \hat{\sigma}(z) = \int_0^z E(z-z')\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}z'}\mathrm{d}z', \quad E(z) = E_\infty + \sum_{\alpha=1}^{n} E_\alpha \mathrm{e}^{-\lambda_\alpha z}, \quad E_\infty = E_0 - \sum_{\alpha=1}^{n} E_\alpha. \tag{13}$$

Although the pattern followed in the construction of the above model slightly differs from that presented in Valanis (1970, 1980), the model rests upon the same constitutive equations. In particular, the choice $\kappa = 0$ furnishes the one-dimensional specialization of the OVET put forward in Valanis (1970), while the choice $\kappa = 1$ and $P(\varepsilon_{ij}) = 1$ yields the one-dimensional specialization of the IVET advanced in Valanis (1980). If in addition to this last choice, the number of inner variables n is taken to be infinite, one obtains the one-dimensional specialization of the more recent developments of endochronic theory such as proposed in Valanis and Lee (1984).

3. THE BEHAVIOUR OF AN ENDOCHRONIC POYNTING-THOMSON MODEL WITH RESPECT TO κ

We shall now analyse how the stress-response of the endochronic model presented in this work is subject to κ. So as to keep the analysis at a minimum, we choose for this purpose an endochronic model endowed with only one inner variable. It is evident that such a study can only be undertaken if the endochronic model is formulated in an incremental way, since the stress-response of the model constructed in the previous section is no longer explicitly, as in the OVET, but implicitly determined, the reason for it lying in the definition of ζ in terms of both the strain as well as the stress, i.e. of both the independent as well as the dependent variable, ε and σ, respectively. We note in passing, that this additional complication in the implementation of the endochronic theory which arises from the use of such a strain path is obviously one major drawback of the IVET with respect to the OVET.

We now turn our attention to the stress-response (1) assumed in Section 2, choose $n = 1$, and differentiate

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon + \rho B\,\mathrm{d}x. \tag{14}$$

In order to arrive at a differential equation in σ and ε only, we first invoke the endochronic evolution equation (10), which we use for $n = 1$. Substituting in (14), we obtain thus

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon - (\rho\eta B\varepsilon + \rho\lambda Bx)\,\mathrm{d}z. \tag{15}$$

The inner variable entering (15) must be eliminated and expressed in terms of σ and ε by using (1). Substitution in (15) yields then

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon - [(\rho\eta B - \lambda E_0)\varepsilon + \lambda\sigma]\,\mathrm{d}z. \tag{16}$$

In accordance with the definitions given in Section 2 of the material parameters E_α and E_∞, (16) may be expressed as

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon - \lambda(E_\infty\varepsilon - \sigma)\,\mathrm{d}z \tag{17}$$

which formally resembles a Poynting–Thomson (or 3-parameter) model. We need now to eliminate $\mathrm{d}z$ in (17) and express it in terms of $\mathrm{d}\varepsilon$ and $\mathrm{d}\sigma$. Accordingly, we substitute $\mathrm{d}\zeta$ in $(7)_2$ by $(6)_2$ and obtain with the help of (4)

$$\mathrm{d}z = \frac{P(\varepsilon_i)}{f(\zeta)}\left|\mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma\right|. \tag{18}$$

It is now convenient to eliminate the absolute value sign therein prior to its substitution in (17). To this end, we must discern among the three possible cases to which we will refer as cases a, b and c through the rest of this paper.

Case a

$$\text{if } \mathrm{d}\varepsilon > \frac{\kappa}{E_0}\mathrm{d}\sigma \Rightarrow \left|\mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma\right| = \mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma. \tag{19}$$

Case b

$$\text{if } \mathrm{d}\varepsilon < \frac{\kappa}{E_0}\mathrm{d}\sigma \Rightarrow \left|\mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma\right| = -\left(d\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma\right). \tag{20}$$

Case c

$$\text{if } \mathrm{d}\varepsilon = \frac{\kappa}{E_0}\mathrm{d}\sigma \Rightarrow \left|\mathrm{d}\varepsilon - \frac{\kappa}{E_0}\mathrm{d}\sigma\right| = 0 \Leftrightarrow \mathrm{d}\zeta = 0 \Leftrightarrow \mathrm{d}z = 0. \tag{21}$$

Let us first consider the rather trivial case c. As a consequence of $\mathrm{d}z = 0$, (2) leads to $\mathrm{d}\dot{x}_z = 0$. We conclude furthermore from (14)

$$\mathrm{d}\dot{x}_z = 0 \Leftrightarrow \mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon. \tag{22}$$

Comparing now $(22)_2$ with $(21)_1$, we infer the following implications:

$$\mathrm{d}\varepsilon = \frac{\kappa}{E_0}\mathrm{d}\sigma \Rightarrow (1-\kappa)\,\mathrm{d}\varepsilon = 0 \Rightarrow (\kappa \neq 1 \Rightarrow \mathrm{d}\varepsilon = 0). \tag{23}$$

Thus, we conclude that for $0 < \kappa < 1$ the slope of the σ–ε curve cannot possibly assume the value E_0/κ (this will be corroborated in Fig. 1). In particular, for $\kappa = 1$ the possibility of obtaining $\mathrm{d}\varepsilon \neq 0$ and $\mathrm{d}z = 0$ cannot be precluded as in the case of the OVET.

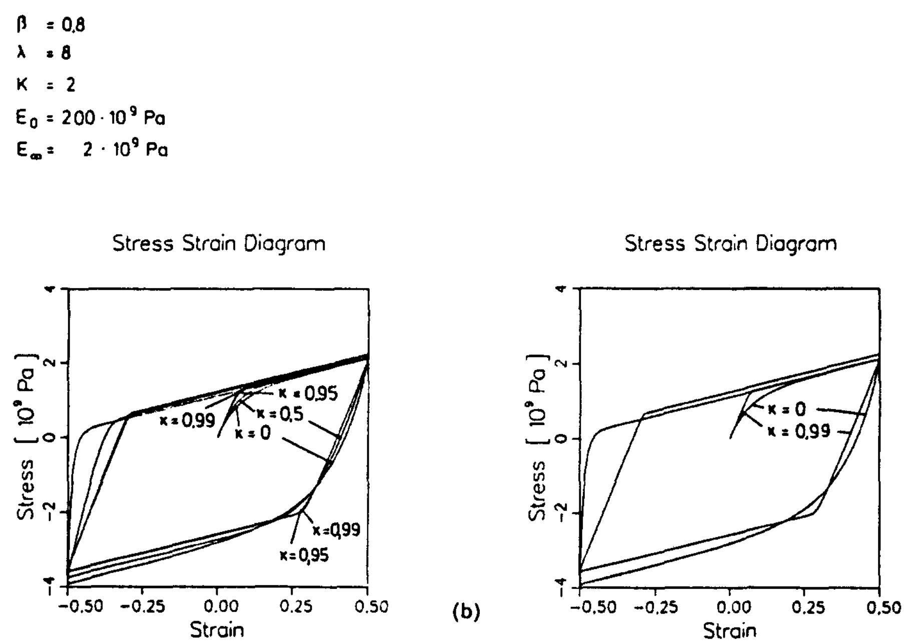

Fig. 1. (a) Stress-response of the endochronic Poynting–Thomson model. (b) Comparison of the limit cases.

As for the other two cases, a and b, we note that they do not necessarily coincide with loading and unloading respectively, unless, of course, we set $\kappa = 0$.

By substituting (19) or (20) in (18) according to whichever case applies, and then replacing this expression in (17), we obtain, after rearranging terms, the desired incremental stress-response of the endochronic Poynting–Thomson model,

$$d\sigma = E_0\left[\frac{E_0 f(\zeta) \pm P(\varepsilon_i)\lambda(E_\infty \varepsilon - \sigma)}{E_0 f(\zeta) \pm \kappa P(\varepsilon_i)\lambda(E_\infty \varepsilon - \sigma)}\right] d\varepsilon \tag{24}$$

where the + sign holds for case a and the − for case b.

As a first observation concerning (24), we remark that the slope of the σ–ε curve starting from the origin (0, 0), i.e. of the particular solution of (24) satisfying the initial conditions $\varepsilon = 0 \Rightarrow \sigma = 0$, will always be equal to E_0 at the origin, irrespective of the value chosen for κ. Indeed, by setting $\varepsilon = 0$ and $\sigma = 0$ in (24), it follows that

$$\forall\kappa : 0 \leqslant \kappa \leqslant 1 : \left.\frac{d\sigma}{d\varepsilon}\right|_0^+ = \left.\frac{d\sigma}{d\varepsilon}\right|_0^- = E_0. \tag{25}$$

We now consider the two limit cases $\kappa = 0$ and $\kappa = 1$ separately. For the limit case $\kappa = 0$, corresponding to the OVET, (24) yields

$$\text{if } \kappa = 0 \Rightarrow \begin{cases} d\varepsilon > 0 \Rightarrow \left.\dfrac{d\sigma}{d\varepsilon}\right|^+ = E_0\left[1 - \dfrac{\lambda}{E_0}\dfrac{P(\varepsilon)}{f(\zeta)}(E_\infty \varepsilon - \sigma)\right] & (26) \\ d\varepsilon < 0 \Rightarrow \left.\dfrac{d\sigma}{d\varepsilon}\right|^- = E_0\left[1 + \dfrac{\lambda}{E_0}\dfrac{P(\varepsilon)}{f(\zeta)}(E_\infty \varepsilon - \sigma)\right] & (27) \end{cases}$$

which, as indicated above, correspond to loading and unloading. As was to be expected, summation of (26) and (27) yields the relation obtained for the OVET in Rivlin (1981).

We now turn to the other limit case, namely $\kappa = 1$, corresponding to the IVET. By setting $\kappa = 1$ in (24) we obtain a somewhat perplexing result for the particular solution satisfying the initial conditions $\varepsilon = 0 \Rightarrow \sigma = 0$,

$$\kappa = 1 \Rightarrow d\sigma = E_0\, d\varepsilon \Rightarrow \sigma = E_0 \varepsilon \tag{28}$$

which is *independent* of the direction of the strain-increment. This means that the limit case $\kappa = 1$ of the endochronic Poynting–Thomson model, which makes use of precisely a plastic strain-path, degenerates into the elastic law $(28)_3$. (This statement will also hold if we let $\kappa \to 1$, since the right side of (24) is continuous in κ at $\kappa = 1$). Furthermore, this result is valid irrespective of the form chosen for the material functions $f(\zeta)$ and $P(\varepsilon_i)$, as we have not taken them into account in the derivation of (24). The reasons accounting for this apparently paradoxical result will be examined in the next two sections, and will be given more consideration in a forthcoming paper. [The reader may be referred to Fazio (1987), where this question is treated in further detail.]

If we now wish to obtain the stress-response of the endochronic Poynting–Thomson model, we must solve the initial value problem consisting in finding a solution of (24) satisfying the initial conditions $\varepsilon = 0 \Rightarrow \sigma = 0$. Evidently, the solution of this problem requires the knowledge of the specific form chosen for the material functions $f(\zeta)$ and $P(\varepsilon_i)$ entering (24). Accordingly, we follow Valanis in Valanis (1971b) for the sake of simplicity and choose the former to be linear and the latter to be constant, i.e.

$$f(\zeta) = 1+\beta\zeta, \quad \beta \geqslant 0 \tag{29}$$

$$P(\varepsilon_t) = K, \quad K > 0 \tag{30}$$

where β and K are dimensionless material parameters.

We must now express ζ appearing in (29) in terms of ε and σ. To this end, we substitute (30) and (4) in $(6)_2$ and obtain after integration

$$\zeta = \pm\left[K(\varepsilon-\varepsilon_0)-\frac{\kappa}{E_0}(\sigma-\sigma_0)\right]+\zeta_0 \tag{31}$$

where ζ_0 denotes the initial value of ζ corresponding to the initial values ε_0 and σ_0, and where it is understood that the + sign holds for case a and the − for case b. We will follow this convention in the remaining part of this paper, so that no ambiguity should arise when encountering a $\pm$ sign in the text.

We now substitute first (31) in (29), and then substitute this expression along with (30) in (24) to obtain, after rearrangement of terms, the following non-autonomous differential equation in normal form

$$\frac{d\sigma}{d\varepsilon} = \left(\frac{a_1\varepsilon+b_1\sigma+c}{a_2\varepsilon+b_2\sigma+c}\right)E_0 \equiv F(\sigma, \varepsilon, \kappa) \tag{32}$$

where we have introduced for brevity the following continuously differentiable functions in κ (the first being a constant):

$$a_1 = \pm(\beta E_0+\lambda E_\infty) \tag{33}$$

$$a_2 = \pm(\beta E_0+\kappa\lambda E_\infty) \tag{34}$$

$$b_1 = \pm(-\kappa\beta-\lambda) \tag{35}$$

$$b_2 = \pm\kappa(-\beta-\lambda) \tag{36}$$

$$c = \frac{E_0}{K}(1+\beta\zeta_0)\pm(\kappa\sigma_0-E_0\varepsilon_0). \tag{37}$$

It must be stressed that the solutions of (32) need not be continuous in κ, since $F(\sigma, \varepsilon, \kappa)$ is not continuous in the entire σ–ε plane.

As a first step towards the solution of (32), we reduce this first-order ordinary differential equation to a homogeneous one by means of the standard coordinate transformation given by

$$\mu = \sigma-\sigma_Q \tag{38}$$

$$\xi = \varepsilon-\varepsilon_Q \tag{39}$$

where the pair $(\sigma_Q, \varepsilon_Q)$ represents the solution of the following system of linear equations:

$$a_1\varepsilon+b_1\sigma+c = 0 \tag{40}$$

$$a_2\varepsilon+b_2\sigma+c = 0. \tag{41}$$

Hence

$$\sigma_Q = \left(\frac{a_2 - a_1}{\Delta}\right)c = \pm(\kappa-1)\lambda\frac{E_x}{\Delta}\left[\frac{E_0}{K}(1+\beta\zeta_0) \pm (\kappa\sigma_0 - E_0\varepsilon_0)\right] \tag{42}$$

$$\varepsilon_Q = \left(\frac{b_1 - b_2}{\Delta}\right)c = \pm(\kappa-1)\frac{\lambda}{\Delta}\left[\frac{E_0}{K}(1+\beta\zeta_0) \pm (\kappa\sigma_0 - E_0\varepsilon_0)\right] \tag{43}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (1-\kappa)\beta[(\kappa\beta+\lambda)E_0 - \kappa\lambda E_r]. \tag{44}$$

It is significant that this transformation is inadmissible for the limit case $\kappa = 1$, since $\kappa = 1 \Rightarrow \Delta = 0$, as can be seen in (44).

After substitution of σ and ε by μ and ξ as given by (38) and (39), respectively, we obtain from (32)

$$\frac{d\mu}{d\xi} = \left(\frac{a_1\xi + b_1\mu}{a_2\xi + b_2\mu}\right)E_0. \tag{45}$$

We may choose to represent (45) by an autonomous system of linear differential equations given by

$$\dot{\mu} = (a_1\xi + b_1\mu)E_0 \tag{46}$$

$$\dot{\xi} = (a_2\xi + b_2\mu). \tag{47}$$

It can be shown then (see Hurewicz, 1958) that the asymptotic behaviour of the characteristics of this system (or alternatively, of the solutions of (45)) will only depend on the values of the coefficients of the latter.† Consequently, the asymptotic behaviour of the characteristics of (46) and (47) is not affected by the initial values ε_0, σ_0 and ζ_0, since they do not enter the definitions of a_1, b_1, a_2 and b_2 (although the coordinates ε_Q and σ_Q of the isolated singularity at the origin $\xi = 0$, $\mu = 0$ do depend on these values). This behaviour may be observed in Fig. 1(a, b).

We now return to (45) in which we separate variables by means of the following transformation,

$$v = \frac{\mu}{\xi} \Leftrightarrow \mu = v\xi \Rightarrow \frac{d\mu}{d\xi} = \xi\frac{dv}{d\xi} + v. \tag{48}$$

Upon substituting μ and $d\mu/d\xi$ in (45) by $(48)_2$ and $(48)_3$, respectively, we can eliminate μ therein to obtain

$$\frac{(a_2 + b_2 v)\,dv}{E_0 a_1 + mv - b_2 v^2} = \frac{d\xi}{\xi} \tag{49}$$

where we have introduced for brevity

$$m = E_0 b_1 - a_2 = \pm\{-[(\kappa+1)\beta + \lambda]E_0 - \kappa\lambda E_r\}. \tag{50}$$

The left side of (49) can be easily integrated by twofold substitution, provided that $b_2 \neq 0$, which according to its definition given by (36), will be the case unless $\kappa = 0$. We thus obtain

† The roots of the characteristics equation of (46) and (47) are found to be real, provided that $E_0 > E_r$.

after transforming the variables v, μ and ξ back to the original ones, σ and ε, the following general solutions of (32) when $0 < \kappa < 1$

$$(E_0 b_1 + a_2)\frac{1}{\sqrt{\Delta_D}}\arctan\left\{\frac{1}{\sqrt{\Delta_D}}\left[-2b_2\left(\frac{\sigma-\sigma_Q}{\varepsilon-\varepsilon_Q}\right)+m\right]\right\}$$
$$= \ln\sqrt{E_0 a_1(\varepsilon-\varepsilon_Q)^2 + m(\sigma-\sigma_Q)(\varepsilon-\varepsilon_Q) - b_2(\sigma-\sigma_Q)^2} + C,$$
$$\forall\kappa: 0 < \kappa < 1, \quad \text{when } \Delta_D > 0 \tag{51}$$

and

$$(E_0 b_1 - 3a_2)\frac{1}{\sqrt{-\Delta_D}}\operatorname{arg\,tanh}\left\{\frac{1}{\sqrt{-\Delta_D}}\left[-2b_2\left(\frac{\sigma-\sigma_Q}{\varepsilon-\varepsilon_Q}\right)+m\right]\right\}$$
$$= \ln\sqrt{E_0 a_1(\varepsilon-\varepsilon_Q)^2 + m(\sigma-\sigma_Q)(\varepsilon-\varepsilon_Q) - b_2(\sigma-\sigma_Q)^2} + C,$$
$$\forall\kappa: 0 < \kappa < 1, \quad \text{when } \Delta_D < 0 \tag{52}$$

where C has the meaning of an integration constant and Δ_D is a discriminant given by

$$\Delta_D = -4a_1 b_2 E_0 - m^2 = -\beta[3\kappa\beta + (1+2\kappa)\lambda]E_0 + \kappa\lambda(\beta+\lambda^2)E_\infty \tag{53}$$

which as one sees, is independent of the direction of the strain increment.

We consider lastly the limit case $\kappa = 0$, for which (49) is reduced to the following singular case

$$\frac{a_2\,\mathrm{d}v}{E_0 a_1 + mv} = \frac{\mathrm{d}\xi}{\xi}. \tag{54}$$

Integration by substitution yields

$$\sigma = \sigma_Q + \frac{\varepsilon-\varepsilon_Q}{m}[-E_0 a_1 + C(\varepsilon-\varepsilon_Q)^{m/a_2}]. \tag{55}$$

We note that only the limit cases $\kappa = 0$ and $\kappa = 1$ lead to an explicit solution of (32) given by (55) and Hooke's Law $(28)_3$, respectively. The σ–ε curve predicted by the model discussed in this section has been plotted in Fig. 1(a, b) for different values of κ. These loci show the stress-response of the endochronic Poynting–Thomson model to a strain history consisting in loading from $\varepsilon = 0$ to $\varepsilon = 0.5$, followed by unloading down to $\varepsilon = -0.5$ and subsequent reloading up to $\varepsilon = 0.5$. (The values chosen for the material constants have been indicated on the diagrams.)

4. THE BEHAVIOUR OF THE ENDOCHRONIC EVOLUTION EQUATIONS WITH RESPECT TO κ

If we recapitulate the general assumptions (1), (2) that have served as the point of departure in the development of the one-dimensional specialization of the endochronic theory, we notice that only the endochronic equations (2) among them depend on the particular value chosen for κ. It seems therefore worthwhile to study the nature of this dependency when κ assumes the value 1 or tends to 1. This will be the objective of this section.

For the sake of generality we will make no restriction from now on the number n of inner variables used in the model considered, which may also be considered to be infinite as in Valanis and Lee (1984). We recall (1) which we differentiate

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon - \rho \sum_{\alpha=1}^{n} B_\alpha\,\mathrm{d}x_\alpha \Leftrightarrow \frac{\mathrm{d}\sigma}{E_0} - \mathrm{d}\varepsilon = \frac{\rho}{E_0}\sum_{\alpha=1}^{n} B_\alpha\,\mathrm{d}x_\alpha. \tag{56}$$

A comparison with the definition of $\mathrm{d}\varepsilon_p$ given by (5) yields

$$\mathrm{d}\varepsilon_p = -\frac{\rho}{E_0}\sum_{\alpha=1}^{n} B_\alpha\,\mathrm{d}x_\alpha, \quad \forall\kappa: 0 \leqslant \kappa \leqslant 1. \tag{57}$$

To obtain $\mathrm{d}x_\alpha$ therein we invoke the endochronic evolution equations (2) and replace $\mathrm{d}z$ by $(7)_2$, which in conjunction with $(4)_2$ yields

$$\mathrm{d}x_\alpha = \frac{P(\varepsilon_i)}{f(\zeta)}|(1-\kappa)\,\mathrm{d}\varepsilon - \kappa\,\mathrm{d}\varepsilon_p|\Omega_\alpha(\hat{\varepsilon}, \hat{\theta}, x_\beta), \quad \forall\kappa: 0 \leqslant \kappa \leqslant 1. \tag{58}$$

Substitution in (57) leads then to

$$\mathrm{d}\varepsilon_p = -\frac{\rho}{E_0}\frac{P(\varepsilon_i)}{f(\zeta)}|(1-\kappa)\,\mathrm{d}\varepsilon - \kappa\,\mathrm{d}\varepsilon_p| \sum_{\alpha=1}^{n} B_\alpha\Omega_\alpha(\hat{\varepsilon}, \hat{\theta}, \hat{x}_\beta). \tag{59}$$

Let us assume now that $\mathrm{d}\varepsilon_p \neq 0$ when $\kappa = 1$ (or $\kappa \to 1$). Then (59) could be converted into

$$\pm f(\zeta_p) = \frac{\rho}{E_0}P(\varepsilon_p)\sum_{\alpha=1}^{n} B_\alpha\Omega_\alpha(\hat{\varepsilon}, \hat{\theta}, \hat{x}_\beta) \quad \text{when } \kappa = 1 \text{ (or } \kappa \to 1) \tag{60}$$

where ζ_p denotes ζ defined in terms of ε_p, i.e. when $\varepsilon_i = \varepsilon_p$, and hence $\mathrm{d}\zeta = P(\varepsilon_p)|\mathrm{d}\varepsilon_p| \equiv \mathrm{d}\zeta_p$. Evidently (60) signifies a constraint condition placed on the endochronic evolution equations when $\kappa = 1$ (or $\kappa \to 1$), which impedes all plastic deformations. Indeed, suppose at least one of the endochronic evolution equations were not identically equal to zero when $\kappa = 1$ (or $\kappa \to 1$) so that as a result of (59) plastic deformations could occur. Clearly, this endochronic evolution equation would then not be uniquely determined on account of the $\pm$ sign appearing in (60), which is, of course, inadmissible. Moreover, this statement is corroborated by the fact that any cycle that could take place in the state-space ε, θ, x_α would violate (60), if any plastic deformations occurred during the cycle, since $f(\zeta)$ was assumed to be a monotonous non-decreasing function (obviously, this last argument will no longer hold if $f(\zeta)$ is chosen to be constant).

We have thus extended the validity of our statement concerning the degeneration of the endochronic theory into Hooke's Law for $\kappa = 1$ (or $\kappa \to 1$) to any number n of inner variables (which may be let grow indefinitely). This result is furthermore independent of the form chosen for the material functions Ω_α, $f(\zeta)$, and $P(\varepsilon_i)$, since we have not taken them into account in the course of our argument.

We close our study of the endochronic evolution equations by inferring the following equivalence from (56) and (2):

$$\text{if } \mathrm{d}z \neq 0 \Rightarrow \forall\kappa: 0 \leqslant \kappa \leqslant 1: \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = E_0 \Leftrightarrow \sum_{\alpha=1}^{n} B_\alpha\Omega_\alpha(\hat{\varepsilon}, \hat{\theta}, \hat{x}_\beta) = 0 \tag{61}$$

which holds irrespective of the direction of the strain-increment. Particularly, (61) indicates that any one-dimensional endochronic model can only then predict at an arbitrary point of the σ–ε diagram elastic unloading, if the foregoing loading at that point was likewise elastic (recall that the sum $\rho\,\Sigma_{\alpha=1}^{n} B_\alpha\,\mathrm{d}x_\alpha$ given by the left side of $(56)_2$ governs the deviation from elastic behaviour). This proves that Valanis' attempt to predict elastic unloading following inelastic loading using z as the independent variable within the thermodynamic framework of his endochronic theory cannot be accomplished without the introduction of an *a priori* loading criterion.

5. THE MOST GENERAL MODEL OF THE IVET

We have hitherto demonstrated that any one-dimensional model of the endochronic theory constructed following Valanis' guidelines degenerates into an elastic law for any number of inner variables, and any particular form chosen for $P(\varepsilon_i)$, $f(\zeta)$ and Ω_r, when we set κ equal to 1, or let κ tend to 1. The question arises then, whether such a result is also verified if we take the stress-response of the IVET for given, in other words, if we postulate it instead of deriving it from thermodynamic considerations. The following theorem answers this question.

Theorem. Let the stress-response of a one-dimensional model be given by

$$\sigma = E_0\varepsilon - \int_0^z G(z-z')\hat{\varepsilon}(z')\,\mathrm{d}z' \tag{62}$$

where $G(z)$ may be arbitrary but is assumed to be continuous in $[0,\infty)$ (such an assumption amounts to postulating the stress-response given by (1) and letting the second term therein take the form of a hereditary integral as in (11)). Then (62) yields Hooke's Law when $\kappa = 1$ (or alternatively when $\kappa \to 1$), i.e. when $\varepsilon_i = \varepsilon_p$ and hence z is defined in terms of ε_p.

In what follows, we shall denote the intrinsic time for this limit case with z_p, i.e. $\mathrm{d}z_p = \mathrm{d}\zeta_p/f(\zeta_p)$.

Proof. The thesis shall be demonstrated by *reductio ad absurdum*, for we will suppose the intrinsic time scale $z = g_{\varepsilon_i}(t)$ to obey

$$\dot{z}(0) = \frac{\mathrm{d}}{\mathrm{d}t}g_{\varepsilon_i}(0) > 0 \tag{63}$$

and show later that z must remain constant when $\kappa = 1$ (or $\kappa \to 1$), which in turn implies, in view of (62), that the model assumed must behave elastically.

As a first step, we calculate the slope of the σ–ε curve by using the chain-rule of differentiation (assuming always $\dot{z}(t) > 0$ at the point considered)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\varepsilon}. \tag{64}$$

The second derivative on the right side therein can be calculated by making use of the chain-rule again, and with the help of the definitions of z and ζ

$$\frac{\mathrm{d}z}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}z}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\varepsilon} = \frac{1}{f(\zeta)}\frac{\mathrm{d}\zeta}{\mathrm{d}\varepsilon} = \frac{P(\varepsilon_i)}{f(\zeta)}\left|1 - \frac{\kappa}{E_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|. \tag{65}$$

The absolute value-bars therein can be eliminated by distinguishing between the cases a and b given by (19) and (20), respectively. Accordingly, we obtain

$$\left.\frac{\mathrm{d}z}{\mathrm{d}\varepsilon}\right|^{\pm} = \pm\frac{P(\varepsilon_i)}{f(\zeta)}\left(1 - \frac{\kappa}{E_0}\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|^{\pm}\right) \tag{66}$$

where the + sign corresponds to case a and the − to case b, as was explained before.

We now turn to the first derivative appearing on the right side of $(65)_2$, which can be obtained by differentiating (62). Prior to this differentiation, we transform (62) as in Section 2 by integration by parts with the initial condition $\hat{\varepsilon}(0) = 0$, and then endochron-parametrize the free terms (which necessitates that (63) hold). Thus, the stress-response assumed can be converted into

$$\sigma = \hat{\sigma}(z) = \int_0^z \bar{E}(z-z')\frac{d\varepsilon}{dz'}dz' \tag{67}$$

where we have introduced a kernel defined as the sum of a constant $\bar{E}_\infty$ plus a primitive $\bar{E}_1(z)$ of $G(z)$, i.e.

$$\bar{E}(z) = \bar{E}_\infty - E_1(z) \tag{68}$$

where

$$\frac{d\bar{E}_1(z)}{dz} = G(z) \tag{69}$$

$$\bar{E}_\infty = E_0 + \bar{E}_1(0). \tag{70}$$

We now differentiate the endochron-parametrized stress-response (67) with the help of Leibniz's rule, and obtain

$$\frac{d\sigma}{dz} = E_0\frac{d\varepsilon}{dz} + \int_0^z G(z-z')\frac{d\varepsilon}{dz'}dz' \tag{71}$$

where we have taken account of (68)–(70). By substituting (66) and (71) in (64) (where in view of the $\pm$ sign appearing in (66), the distinction between cases a and b must also be made) and rearranging terms, we obtain

$$\left.\frac{d\sigma}{d\varepsilon}\right|^{\pm} = E_0\frac{\left(E_0 \pm \dfrac{P(\varepsilon_t)}{f(\zeta)}\displaystyle\int_0^z G(z-z')\frac{d\varepsilon}{dz'}dz'\right)}{\left(E_0 \pm \kappa\dfrac{P(\varepsilon_t)}{f(\zeta)}\displaystyle\int_0^z G(z-z')\frac{d\varepsilon}{dz'}dz'\right)}, \quad \forall\kappa: 0 \leqslant \kappa \leqslant 1. \tag{72}$$

By setting $\kappa = 1$ (or, alternatively, by letting $\kappa \to 1$), (72) yields

$$\left.\frac{d\sigma}{d\varepsilon}\right|^{\pm} = E_0 \Leftrightarrow d\sigma = E_0\, d\varepsilon. \tag{73}$$

This local form of Hooke's Law implies, according to the definition of $d\varepsilon_p$, that $d\varepsilon_p$ and hence z_p must vanish for $\kappa = 1$ (or $\kappa \to 1$) at the point considered. This holds particularly for the beginning of the deformation process (i.e. for $t = 0$), and consequently contradicts the assumption (63) for $\kappa = 1$ (or $\kappa \to 1$). Hence, z_p will remain equal to zero as ε increases, as a consequence of which (62) is reduced to the form

$$\sigma = E_0\varepsilon. \tag{74}$$

QED

Remark 1 on the theorem. Had we taken (67) as the assumed stress-response (which is tantamount to postulating (13) as such and replacing the kernel therein by an arbitrary one), then only the local form of Hooke's Law $(73)_2$ could hold, since it is obvious that (67) cannot possibly describe (74) when z is constant.

Remark 2 on the theorem. The statement proved in this theorem holds now not only for any particular form chosen for $f(\zeta)$ and $P(\varepsilon_t)$ but also for any form chosen for the kernel $G(z)$ in (62). We may therefore conclude that the degeneration of the one-dimensional case of the endochronic theory for $\kappa = 1$ (or $\kappa \to 1$) can only be ascribed to the use of an inappropriate intrinsic time scale, namely $z_p = g_{\varepsilon_p}(t)$.

Remark 3 on the theorem. The reasons for which the endochronic theory degenerates into Hooke's Law for $\kappa = 1$ (or $\kappa \to 1$) can be elucidated now, if we recall that $z_p = g_{\varepsilon_p}(t)$, contrary to $z = g_{\varepsilon_i}(t)$ for $\kappa < 1$, cannot exhibit the property (63), since $\mathrm{d}\sigma/\mathrm{d}\varepsilon|^{\pm} = E_0$ for any endochronic model, and consequently $\dot{z}_p(0) = 0$.

Remark 4 on the theorem. No yielding phenomenon is susceptible of being described by the present one-dimensional specialization of the endochronic theory for which Valanis claims the contrary for $\kappa = 1$ in Valanis (1980) and Valanis and Lee (1984). Indeed, were any yielding feasible after some finite elastic response, then $\mathrm{d}\varepsilon_p > 0 \Rightarrow \dot{z}_p > 0$ at that point. However, this is ruled out by (73).

Remark 5 on the theorem. We can derive as a special case of (72) the relation laid down in Rivlin (1981) corresponding to the OVET by simply setting $\kappa = 0$ (which implies $\varepsilon_i = \varepsilon$) and taking $\bar{E}(z) = E(z)$.

Corollary. Valanis' attempt to predict an elastic unloading following inelastic loading without the introduction of an *a priori* loading criterion can only be achieved, within the context of a one-dimensional endochronic model irrespective of κ, if the aforementioned model is reduced to an elastic law.

The proof is straightforward. We sum $\mathrm{d}\sigma/\mathrm{d}\varepsilon|^{+}$ and $\mathrm{d}\sigma/\mathrm{d}\varepsilon|^{-}$ given by (72) and rearrange terms so as to derive a relationship between both slopes at a given point of the σ–ε diagram

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|^{+} = \frac{2E_0 - \left.\dfrac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|^{-}\left(1 - \dfrac{\kappa}{E_0}\dfrac{P(\varepsilon_i)}{f(\zeta)}\displaystyle\int_0^z G(z-z')\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}z'}\,\mathrm{d}z'\right)}{1 + \dfrac{\kappa}{E_0}\dfrac{P(\varepsilon_i)}{f(\zeta)}\displaystyle\int_0^z G(z-z')\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}z'}\,\mathrm{d}z'}, \quad \forall\kappa : 0 \leqslant \kappa \leqslant 1. \tag{75}$$

This relationship between the two slopes univocally determines one of them if the other is prescribed. In particular, (75) implies that for any value of κ an elastic unloading at an arbitrary point of the σ–ε curve can only be described if the foregoing loading at that point was likewise elastic. Mathematically stated

$$\forall\kappa : 0 \leqslant \kappa \leqslant 1 : \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|^{-} = E_0 \Leftrightarrow \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|^{+} = E_0. \tag{76}$$

At a global level, the above statement implies that the present model can only then predict an elastic unloading at every point of the σ–ε diagram as sought by Valanis, when the model is reduced to Hooke's Law. This statement, which was made in the previous section, holds now for any value of κ, and any particular form chosen for the material functions $f(\zeta)$, $P(\varepsilon_i)$ and $G(z)$.

Remark 1 on the corollary. By setting $\kappa = 0$ in (75) we obtain as a special case corresponding to the OVET the relation given in Rivlin (1981).

Remark 2 on the corollary. The previous observations made on the limit case $\kappa = 1$ (or $\kappa \to 1$) do not exclude the possibility of approximating the slope of the σ–ε curve on the onset of unloading by setting κ sufficiently close (but not equal) to 1, as a perusal of Fig. 1(a, b) suggests. This is justified by the fact that $\mathrm{d}\sigma/\mathrm{d}\varepsilon|^{+}$ approaches E_0 considerably slower with κ than $\mathrm{d}\sigma/\mathrm{d}\varepsilon|^{-}$ as explained in Fazio (1987). This promising possibility has already been successfully exploited in the literature to describe the mechanical behaviour of metals (see Wu and Yang, 1983).

6. CONCLUDING REMARKS

It has been shown that no one-dimensional endochronic model is capable of predicting an elastic unloading following inelastic loading, far less yielding, as maintained in Valanis (1980) as well as in Valanis and Fan (1983, p. 790). Indeed, where Valanis asserts to have derived a yielding condition and loading criteria from his theory, our one-dimensional specialization of the latter yields an elastic law. This represents a degeneration of the theory, the nature of which lies in the use of an inadequate intrinsic time scale, namely, one based on the plastic strain. It has been shown that such an intrinsic time scale is unsuitable for the thermodynamic setting of the theory as well as for the stress-response of the most general one-dimensional endochronic model.

It should be pointed out however, that the foregoing criticisms do not apply to the use of an intrinsic time scale based on the plastic strain such as is done in Watanabe and Atluri (1986). These authors *retain* the notion of a yield surface and loading criteria and *assume* the stress-response to be given by a hereditary integral in the form of (67), in which the total strain is replaced by the plastic strain. This represents one form of the stress-response pretendedly derived for the IVET in Valanis (1980). It is in particular against the claim of having derived such a stress-response from any thermodynamic considerations that the criticisms of the present paper are raised. [We note that Valanis continues to sustain this claim in more recent papers such as Valanis and Lee (1984). We quote the beginning of the Appendix A of the latter: "The theory of course continues to have its foundation in irreversible thermodynamics."]

Although the arguments expounded in this paper are not susceptible of being extrapolated to the three-dimensional case, it will be demonstrated in a forthcoming paper, that the results presented in this work hold also in the three-dimensional case [which is mathematically much more involved; the reader may be referred to Fazio (1987) where a detailed account of this treatment may be found].

Despite the emphasis we have laid on the limitations of the endochronic theory, we believe its advantages should be duly appraised. To be more precise, the endochronic theory allows one to approximate the stress-response of conventional plasticity theories with only one set of constitutive equations. Here lies Valanis' remarkable merit.

Acknowledgement—The author wishes to express his gratitude towards the Deutsche Akademische Austauschdienst (DAAD) for having sponsored this research under a long-term grant.

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